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BY

RUPERT G. MILLER, JR.

TECHNICAL REPORT NO. 88
April 15, 1963

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(NR-342-022)
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1. Introduction

Let $\{X_t\}$, $t \in T = [0,\infty)$, be an irreducible Markov chain in continuous time with state space $I = \{0,1,2,\ldots\}$. The stationary transition probability matrix $P(t) = (p_{ij}(t))$ is assumed to be measurable and satisfy

$$(1.1)\qquad \begin{aligned} & p_{ij}(t) \geq 0, \ \sum_j p_{ij}(t) \leq 1, && i, j \in I \ , \\ & P(t+s) = P(t)\, P(s), \ P(0+) = I \ , \end{aligned}$$

for all $t, s \in T$. In addition, the states are assumed to be stable; i.e.,

$$(1.2)\qquad \begin{aligned} & 0 > p_{ii}'(0) = \lim_{t\downarrow 0} \frac{p_{ii}(t)-1}{t} = q_{ii} = -q_i > -\infty \ , && i \in I \ , \\ & 0 < p_{ij}'(0) = \lim_{t\downarrow 0} \frac{p_{ij}(t)}{t} = q_{ij} < +\infty \ , && i \neq j \in I \ . \end{aligned}$$

The matrix $Q = (q_{ij})$ is called the Q-matrix or infinitesimal generator matrix of the process, and it is assumed to be conservative, i.e., $\sum_j q_{ij} = 0$, $i \in I$. For simplicity, this type of Markov chain will be referred to as a simple continuous time Markov chain (SCMC). A thorough treatise on the properties of a SCMC is contained in [1].

In [8], [9] the solutions to the equations $y\, Q = 0$ were investigated. These stationarity equations are obtained by setting the

derivatives equal to zero in the forward Kolmogorov equations $P'(t) = P(t)Q$, and are the continuous time analog of the stationarity equations $xP = x$ for a discrete time Markov chain (with stationary one-step transition probability matrix $P = (p_{ij})$). In particular, it was shown that, under the minimality assumption described below, a NSC for the SCMC to be positive recurrent is for the equations $yQ = 0$ to have a convergent, positive solution $y = (y_0, y_1, y_2, \ldots)$. The solution is unique (up to a multiplicative constant):

$$y_i = \pi_i = \lim_{t \to \infty} p_{ii}(t) \quad , \qquad i \in I \quad . \tag{1.3}$$

In [9] $yQ = 0$ was also shown to have a unique, positive solution in a null recurrent chain, and for any recurrent chain (positive or null) the relationship between the unique stationary measures of the SCMC and its imbedded discrete time Markov chain was obtained.

The analogous stationarity theorem for positive recurrent chains in discrete time is due to Foster [4] with an earlier, less general version being given by Feller [3], p. 325 (see also Chung [1], p. 33). Foster also gives three additional theorems on conditions for recurrence, ergodicity, etc. in a discrete time chain. The purpose of this paper is to extend these additional theorems to continuous time.

The minimality assumption referred to above which is necessary for the validity of the preceding theorems in continuous time is:

Minimality Assumption: The SCMC is uniquely defined by its Q-matrix; i.e., the minimal process is an honest process (see [1] for details). It will be necessary to impose this assumption in Theorem 2 below but

will not be needed in Theorem 1. It excludes from consideration those processes which can explode to $+\infty$ in finite time. Various necessary and sufficient conditions on the Q-matrix for the minimality assumption to hold have been derived and can be found elsewhere. For reference, see Chung [1] and Reuter [10].

2. Results

In the proofs of this section it will be necessary to refer to the imbedded chain of the SCMC. This is the irreducible (but possibly periodic) discrete time Markov chain $\{X_n\}$, $n = 0,1,2,\cdots$, with stationary transition probability matrix $P = (p_{ij})$ where

$$(2.1) \qquad \begin{aligned} p_{ij} &= q_{ij}/q_i \quad , \qquad && i \neq j \in I \quad , \\ p_{ii} &= 0 \quad , && i \in I \quad . \end{aligned}$$

The imbedded Markov chain $\{X_n\}$ simply records the sequence of states through which the SCMC passes without regard to the amount of time required for the transitions.

<u>Theorem</u> 1: (a) The SCMC $\{X_t\}$ is recurrent if there exists a sequence $z = (z_0, z_1, z_2, \ldots)$ such that (i) $z_n \to +\infty$ as $n \to +\infty$ and (ii) $Qz \leq 0$ except for the first coordinate.

(b) A NSC for the SCMC $\{X_t\}$ to be non-recurrent is that there exist a bounded non-constant sequence $z = (z_0, z_1, z_2, \ldots)$ such that $Qz = 0$ except for the first coordinate.

<u>Proof</u>: The proofs of (a) and (b) are immediate and can be given together. The system of inequalities or equalities $Qz \leq 0$, $= 0$ can be written as

$$\sum_{j=0, \neq i}^{\infty} q_{ij} z_j \leq, = q_i z_i \quad , \qquad i \neq 0 \in I \tag{2.2}$$

Division by q_i yields

$$\sum_{j=0}^{\infty} p_{ij} z_j \leq, = z_i \quad , \qquad i \neq 0 \in I \tag{2.3}$$

where $P = (p_{ij})$ is the transition matrix of the imbedded chain. Under the conditions on z in (a) the system of inequalities (2.3) implies recurrence for the imbedded chain by Theorem 5 of [4]. Similarly, under the conditions on z in (b) the system of equations (2.3) is a NSC for the transience of the imbedded chain by Theorem 4 of [4]. But the recurrence or transience of the imbedded chain is identical to the recurrence or non-recurrence, respectively, of the SCMC. Recurrence is independent of the time component. ||

The term "non-recurrent" rather than "transient" is used here in dealing with a SCMC because of the two possible types of path function behavior. A SCMC can be non-explosive (i.e., satisfy the minimality assumption) but have transient states in the sense that a return to each has probability less than one, or it can be explosive and reach $+\infty$ in finite time with positive probability. In both cases the states of the imbedded chain are transient.

Note that it is not necessary to impose the minimality assumption in Theorem 1. The fact that the imbedded chain has not been defined beyond the first infinity does not cause any difficulty. Should $Qz = 0$ not have a bounded non-constant solution or $Qz \leq 0$ have a solution whose coordinates tend to $+\infty$, the imbedded chain is recurrent, and

by necessity the SCMC is uniquely defined. Should $Qz = 0$ have a bounded non-constant solution, the imbedded chain is transient; the SCMC is then either explosive or non-explosive and transient.

This theorem, particularly part (b), is motivated by the following consideration. Let f_{i0} be the probability that if the SCMC (or its imbedded chain) starts in state i, it reaches state 0 eventually. If f_{00} is defined to be 1, then the f_{i0} satisfy the equations

$$\sum_{j=0}^{\infty} p_{ij} f_{j0} = f_{i0} , \qquad i \neq 0 \in I . \tag{2.4}$$

In a recurrent chain $f_{i0} \equiv 1$, but in a transient chain $f_{i0} \not\equiv 1$, so the f_{i0} constitute a bounded, non-constant solution to (2.4).

In an earlier paper [6] Karlin and McGregor extended Foster's Theorems 4 and 5 to birth and death processes. Utilizing the special structure of these processes they also established necessity in part (a). This does not seem to be true in general (see [4]).

Theorem 2: Under the minimality assumption a NSC that the SCMC be positive recurrent is that the inequalities

$$\sum_{j=0}^{\infty} q_{ij} z_j \leq -1 , \qquad i \neq 0 \in I , \tag{2.5}$$

(i.e., $Qz \leq -1$ except for the first coordinate) have a non-negative solution z which satisfies

$$\sum_{j=1}^{\infty} q_{0j} z_j < +\infty \tag{2.6}$$

(i.e., $|(Qz)_0| < +\infty$).

Proof: (Necessity) Let m_{i0} be the expected time it takes the SCMC to reach state 0 from state $i \neq 0 \in I$; $m_{00} = 0$. For a positive recurrent SCMC $m_{i0} < +\infty$. These expected first-passage times satisfy the equations

$$m_{i0} = \frac{1}{q_i} + \sum_{j=0}^{\infty} p_{ij} m_{j0} \quad , \qquad i \neq 0 \in I \quad , \tag{2.7}$$

where the first term on the right is the expected length of time spent in state i and the second term is the expected time to reach 0 after the process leaves state i. Multiplication of (2.7) by q_i and rearrangement of terms yields (2.5) with equality for $z_j = m_{j0}$.

Since the chain is positive recurrent, the mean recurrence time to state 0 is finite; i.e.,

$$\frac{1}{q_0} + \sum_{j=0}^{\infty} p_{0j} m_{j0} < +\infty \quad , \tag{2.8}$$

which implies (2.6).

(Sufficiency) Rearrangement of terms in (2.5) gives

$$\sum_{j=0}^{\infty} p_{ij} z_j \leq z_i - \frac{1}{q_i} \quad , \qquad i \neq 0 \in I \quad . \tag{2.9}$$

Without loss of generality assume $z_0 = 0$. Consider the iterative inequality obtained by applying $P^n = (p_{ij}^{(n)})$ to z:

$$\sum_{j=0}^{\infty} p_{ij}^{(n)} z_j = \sum_{k=0}^{\infty} p_{ik}^{(n-1)} \sum_{j=0}^{\infty} p_{kj} z_j$$

$$(2.10) \qquad \leq \sum_{k=1}^{\infty} p_{ik}^{(n-1)}(z_k - \frac{1}{q_k}) + p_{i0}^{(n-1)} \sum_{j=0}^{\infty} p_{0j} z_j$$

$$= \sum_{k=0}^{\infty} p_{ik}^{(n-1)} z_k - \sum_{k=0}^{\infty} p_{ik}^{(n-1)} \frac{1}{q_k} + p_{i0}^{(n-1)}(\frac{1}{q_0} + \lambda)$$

where $\lambda = \sum_{j=0}^{\infty} p_{0j} z_j < +\infty$ by (2.6).

The (n-1)-fold interation of this inequality produces the following inequality:

$$(2.11) \qquad 0 \leq \sum_{j=0}^{\infty} p_{ij}^{(n)} z_j \leq \sum_{k=0}^{\infty} p_{ik} z_k - \sum_{v=1}^{n-1} \sum_{k=0}^{\infty} p_{ik}^{(v)} \frac{1}{q_k}$$

$$+ (\frac{1}{q_0} + \lambda) \sum_{v=1}^{n-1} p_{i0}^{(v)} \quad .$$

The series $\sum_{v=1}^{\infty} \sum_{k=0}^{\infty} p_{ik}^{(v)}/q_k$ is divergent by the minimality assumption since it is the expected time required to make an infinite number of transitions after leaving state i. For a rigorous proof see Theorem II. 19.1, Corollary 1, of [1]. But this means that $\sum_{v=1}^{\infty} p_{i0}^{(v)}$ must be divergent in order to preserve the non-negativity in (2.11). Hence, the SCMC is recurrent.

To establish positive recurrence sum the inequality (2.10) for $n = 2,\ldots,N+1$.

$$(2.12) \qquad \sum_{n=2}^{N+1} \sum_{j=0}^{\infty} p_{ij}^{(n)} z_j \leq \sum_{n=1}^{N} \sum_{k=0}^{\infty} p_{ik}^{(n)} z_k - \sum_{n=1}^{N} \sum_{k=0}^{\infty} p_{ik}^{(n)} \frac{1}{q_k}$$

$$+ (\frac{1}{q_0} + \lambda) \sum_{n=1}^{N} p_{i0}^{(n)} \quad .$$

Rearrangement and cancellation produces

$$\sum_{k=0}^{\infty}\left(\sum_{n=1}^{N} p_{ik}^{(n)}\right)\frac{1}{q_k} \le \sum_{k=0}^{\infty} p_{ik}z_k - \sum_{k=0}^{\infty} p_{ik}^{(N+1)}z_k + \left(\frac{1}{q_0}+\lambda\right)\sum_{n=1}^{N} p_{i0}^{(n)}$$

(2.13)

$$\le \sum_{k=0}^{\infty} p_{ik}z_k + \left(\frac{1}{q_0}+\lambda\right)\sum_{n=1}^{N} p_{i0}^{(n)} \quad .$$

Divide both sides of (2.13) by $\sum_{n=1}^{N} p_{ih}^{(n)}$ (which is positive for N sufficiently large) for any $h \in I$. As $N \to \infty$ Fatou's lemma gives

$$(2.14) \qquad \sum_{k=0}^{\infty}\left(\lim_{N\to\infty}\frac{\sum_{n=1}^{N} p_{ik}^{(n)}}{\sum_{n=1}^{N} p_{ih}^{(n)}}\right)\frac{1}{q_k} \le \left(\frac{1}{q_0}+\lambda\right)\lim_{N\to\infty}\frac{\sum_{n=1}^{N} p_{i0}^{(n)}}{\sum_{n=1}^{N} p_{ih}^{(n)}} \quad ,$$

the right hand side reducing to a single term since $\sum_{n=1}^{\infty} p_{ih}^{(n)} = +\infty$ in a recurrent chain. These limits exist by the Doeblin ratio limit theorem and have been evaluated by Chung. For a recurrent chain

$$(2.15) \qquad \lim_{N\to\infty}\frac{\sum_{n=1}^{N} p_{ik}^{(n)}}{\sum_{n=1}^{N} p_{ih}^{(n)}} = \frac{{}_{\ell}p^*_{\ell k}}{{}_{\ell}p^*_{\ell h}} \quad ,$$

for any $\ell \in I$, where ${}_{\ell}p^*_{\ell k}$ is the expected number of visits to state k between visits to state ℓ (${}_{\ell}p^*_{\ell\ell} = 1$). (For reference see [1], Sec. I.9). From (2.14) and (2.15)

$$(2.16) \qquad \sum_{k=0}^{\infty} {}_{\ell}p^*_{\ell k}\frac{1}{q_k} \le \left(\frac{1}{q_0}+\lambda\right) {}_{\ell}p^*_{\ell 0} < +\infty \quad .$$

Derman [2] showed that for a recurrent chain the ${}_{\ell}p^*_{\ell i}$, $i = 0,1,2,\ldots,$ constitute the unique (except for a multiplicative constant) positive solution of the equations $xP = x$. In a recurrent chain the unique solutions of $yQ = 0$ and $xP = x$ are related by $y_i = x_i/q_i$ (see Theorem 3 of [9]). Thus, by (2.16) $y_i = {}_{\ell}p^*_{\ell i}/q_i$ is a positive, convergent solution to $yQ = 0$ so the SCMC is positive recurrent (by Theorem 1 of [9]). ||

The motivation for this theorem is clearly contained in the necessity part of the proof where (2.5) holds with equality for $z_j = m_{j0}$. That the equalities can be replaced with inequalities in the sufficiency condition is a trivial bonus of the proof.

The minimality assumption is essential for the validity of the sufficiency part of the theorem. For a counter-example without it take a birth and death process with

$$(2.17) \qquad \mu_0 = 0 \ , \quad \sum_{n=0}^{\infty} \rho_n < +\infty \ , \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n \rho_n} < +\infty \ ,$$

where $\rho_n = \lambda_0 \lambda_1 \cdots \lambda_{n-1}/\mu_1 \cdots \mu_n$, $n = 1,2,\ldots,\rho_0 = 1$. Such a birth and death process is explosive (see [5]). However,

$$(2.18) \qquad \begin{aligned} z_0 &= 0 \ , \\ z_{n+1} &= z_1 \sum_{v=0}^{n} \frac{\lambda_0}{\lambda_v \rho_v} - \sum_{v=1}^{n} \frac{1}{\lambda_v \rho_v} \sum_{u=0}^{v} \rho_u \ , \qquad n = 1,2,\ldots, \end{aligned}$$

satisfies the equations $\sum_{j=0}^{\infty} q_{ij} z_j = -1$, $i = 1,2,\ldots,$ for any z_1. ($\sum_{j=0}^{\infty} q_{0j} z_j < +\infty$ holds trivially.) For z_1 sufficiently large z_n will be positive for all n since the negative series in (2.18) is convergent.

Kingman [7] proved this theorem for bounded q_i by a different method. Since boundedness of the q_i guarantees the minimality assumption but is not necessary for it to hold, Theorem 2 would constitute an extension of Kingman's result. An application of the sufficiency condition to parallel queues can also be found in [7].

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